Charged Lepton Mass Spectrum and Supersymmetric Yukawaon Model

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Abstract

On the basis of the so-called supersymmetric yukawaon model, it is investigated what form of the superpotential W can lead to the observed charged lepton mass spectrum. A simple form of W can speculate reasonable values of $K(\mu) = (m_e + m_\mu + m_\tau)/(\sqrt{m_e} + \sqrt{m_\mu} + \sqrt{m_\tau})^2$ and $\kappa(\mu) = \sqrt{m_e m_\mu m_\tau}/(\sqrt{m_e} + \sqrt{m_\mu} + \sqrt{m_\tau})^3$ at an energy scale $\mu = \Lambda$.

1 Introduction

If quarks and leptons are fundamental entities in the nature, their mass spectra will obey laws of the nature which have simple forms. Especially, since the charged lepton masses are almost independent of gluon corrections, it will be comparatively easy to identify such laws of the nature. In reference to this problem, for example, we know that an empirical formula [1]

$$m_e + m_\mu + m_\tau = \frac{2}{3}(\sqrt{m_e} + \sqrt{m_\mu} + \sqrt{m_\tau})^2, \tag{1.1}$$

is well-satisfied by the observed charged lepton masses ("pole" mass values). However, usually, masses which we deal with in a conventional mass matrix model are "running" masses $m_i(\mu)$, not pole masses m_i^{pole}. It is also well known that the running masses do not satisfy the relation (1.1). In this paper, we will consider that the relation (1.1) is merely an approximate relation even for the pole masses, so that we will not attempt to derive the relation (1.1) as an exact relation. Instead, we will discuss running masses $m_i(\mu)$ at a high energy scale $\mu = \Lambda$. It is beyond the scope of the present work to explain validity of the mass formula (1.1) with the current experimental accuracy of order 10^{-5}; rather we aim at accuracy of order $\varepsilon \sim 10^{-3}$.

In the present paper, we investigate the charged lepton mass spectrum based on the running masses. Then, for investigating the mass spectrum, it is useful to use the following two quantities

$$K(\mu) \equiv \frac{m_e(\mu) + m_\mu(\mu) + m_\tau(\mu)}{(\sqrt{m_e(\mu)} + \sqrt{m_\mu(\mu)} + \sqrt{m_\tau(\mu)})^2}, \tag{1.2}$$

and

$$\kappa(\mu) \equiv \frac{\sqrt{m_e(\mu)}\sqrt{m_\mu(\mu)}\sqrt{m_\tau(\mu)}}{(\sqrt{m_e(\mu)} + \sqrt{m_\mu(\mu)} + \sqrt{m_\tau(\mu)})^3}, \tag{1.3}$$

instead of the mass ratios $m_e(\mu)/m_\tau(\mu)$ and $m_\mu(\mu)/m_\tau(\mu)$, because $K(\mu)$ and $\kappa(\mu)$ are almost independent of an energy scale μ up to the order of α [2] as we show below: The running masses $m_i(\mu)$ are related to the pole masses m_i^{pole} as

$$m_i(\mu) = m_i^{pole}\left[1 - \frac{\alpha(\mu)}{\pi}\left(1 + \frac{3}{4}\ln\frac{\mu^2}{m_i^2}\right)\right], \tag{1.4}$$

under the one-loop approximation [3]. When we denote $K(\mu)$ and $\kappa(\mu)$ as

$$K(\mu) = K^{pole}(1 + \varepsilon_K(\mu)), \quad \kappa(\mu) = \kappa^{pole}(1 + \varepsilon_\kappa(\mu)), \tag{1.5}$$

where K^{pole} and κ^{pole} are defined by pole masses correspondingly to $K(\mu)$ and $\kappa(\mu)$, respectively, we obtain the deviations $\varepsilon_K(\mu)$ and $\varepsilon_\kappa(\mu)$ as follows:

$$\varepsilon_K(\mu) = \frac{3\alpha(\mu)}{4\pi}\sum_i\left(\frac{\sqrt{m_i}}{\sqrt{m_1} + \sqrt{m_2} + \sqrt{m_3}} - \frac{m_i}{m_1 + m_2 + m_3}\right)\ln\frac{\mu^2}{m_i^2} + O(\alpha^2), \tag{1.6}$$

and

$$\varepsilon_\kappa(\mu) = \frac{3\alpha(\mu)}{4\pi}\sum_i\left(\frac{3}{2}\frac{\sqrt{m_i}}{\sqrt{m_1} + \sqrt{m_2} + \sqrt{m_3}} - \frac{1}{2}\right)\ln\frac{\mu^2}{m_i^2} + O(\alpha^2), \tag{1.7}$$

respectively, where $m_i = m_i(\mu)$. Since we can show

$$\sum_i\left(\frac{\sqrt{m_i}}{\sqrt{m_1} + \sqrt{m_2} + \sqrt{m_3}} - \frac{m_i}{m_1 + m_2 + m_3}\right) = 0, \tag{1.8}$$

$$\sum_i\left(\frac{3}{2}\frac{\sqrt{m_i}}{\sqrt{m_1} + \sqrt{m_2} + \sqrt{m_3}} - \frac{1}{2}\right) = 0, \tag{1.9}$$

the values $\varepsilon_K(\mu)$ and $\varepsilon_\kappa(\mu)$ are independent of μ up to the order of α. The numerical results are as follows:

$$\varepsilon_K(\mu) = \frac{3\alpha(\mu)}{4\pi} \times 0.98938 + O(\alpha^2) = 1.7236 \times 10^{-3} + O(\alpha^2), \tag{1.10}$$

$$\varepsilon_\kappa(\mu) = \frac{3\alpha(\mu)}{4\pi} \times (-9.0562) + O(\alpha^2) = -1.5777 \times 10^{-2} + O(\alpha^2). \tag{1.11}$$

However, the values (1.10) and (1.11) are pure electromagnetic corrections based on Eq.(1.4), and there are corrections from another diagrams, which are dependent on models and slightly show μ-dependence. We show a typical case in a SUSY model with $\tan\beta = 10$ in Table 1, where the input charged lepton mass values are quoted from Ref.[4]. As seen in Table 1, the quantities $K(\mu)$ and $\kappa(\mu)$ are almost insensitive to the energy scale μ. From the results of Table 1, we take

$$\varepsilon_K(\Lambda) \simeq 2.4 \times 10^{-3}, \quad \kappa(\Lambda) \simeq 2.023 \times 10^{-3}, \tag{1.12}$$

Table 1: Energy scale dependence of K and κ in a SUSY model with $\tan\beta = 10$: K and κ for pole masses are given by $K^{pole} = 0.999989 \pm 0.000014$ and $\kappa^{pole} = (2.0633 \pm 0.0001) \times 10^{-3}$.

Scale	$\frac{3}{2}K(\mu)$	$\kappa(\mu)$
$\mu = M_Z$	$1.001879 \pm +0.00002$	$(2.02760^{+0.00015}_{-0.00016}) \times 10^{-3}$
$\mu = 10^3$ GeV	1.00195 ± 0.00002	$(2.02710 \pm 0.00016) \times 10^{-3}$
$\mu = 10^9$ GeV	1.00220 ± 0.00002	$(2.02469 \pm 0.00016) \times 10^{-3}$
$\mu = 10^{12}$ GeV	1.00230 ± 0.00002	$(2.02375^{+0.00017}_{-0.00015}) \times 10^{-3}$
$\mu = 2 \times 10^{16}$ GeV	1.00242 ± 0.00002	$(2.02268^{+0.00016}_{-0.00015}) \times 10^{-3}$

as values which should be explained in this paper.

In reference to this topic, recently, Sumino [5] has proposed an interesting idea that a flavor gauge symmetry induces an effect which exactly cancels the deviation of $\varepsilon_K(\mu)$ at any energy scale μ. However, differently from Sumino's idea, in this paper, we adopt the standpoint that $K^{pole} = 2/3$ is accidental. We consider that our goal is not $K^{pole} = 2/3$, but $K(\Lambda) = \frac{2}{3}(1+\varepsilon(\Lambda))$ with $\varepsilon(\Lambda) \sim 10^{-3}$ from the beginning. ($\varepsilon_K(\mu)$ is deviation of $K(\mu)$ from K^{pole} due to running masses, while $\varepsilon(\Lambda)$ is deviation of $K(\Lambda)$ from the value $2/3$ in the present model.)

In this paper, we try to understand the values (1.12) from a new approach to a mass matrix model, the so-called "yukawaon model" [6, 7]: We regard the Yukawa coupling constants Y_f as "effective" coupling constants Y_f^{eff} in an effective theory, and we consider that Y_f^{eff} originate in vacuum expectation values (VEVs) of new gauge singlet scalars Y_f, i.e.

$$Y_f^{eff} = \frac{y_f}{\Lambda}\langle Y_f\rangle, \tag{1.13}$$

where Λ is a scale of the effective theory which is valid at $\mu \leq \Lambda$, and we assume $\langle Y_f\rangle \sim \Lambda$. We refer the fields Y_f as "yukawaons" hereafter. Note that the effective coupling constants Y_f^{eff} evolve as in the standard SUSY model below the scale Λ, since a flavor symmetry is completely broken at the high energy scale $\mu = \Lambda$ (but the supersymmetry is still unbroken at $\Lambda > \mu > \Lambda_{SUSY} \sim 10^3$ GeV).

In the yukawaon model, the Higgs scalars are the same as ones in a conventional model, i.e. we consider only two Higgs scalars H_u and H_d as an origin of the masses (not as an origin of the mass spectra). For example, we assume an O(3) flavor symmetry [8] and we consider that the yukawaons Y_f are $(\mathbf{3} \times \mathbf{3})_S = \mathbf{1} + \mathbf{5}$ of O(3)$_F$. Then, the would-be Yukawa interactions are given by

$$W_Y = \sum_{i,j} \frac{y_u}{\Lambda}u_i^c(Y_u)_{ij}q_jH_u + \sum_{i,j} \frac{y_d}{\Lambda}d_i^c(Y_d)_{ij}q_jH_d$$

$$+ \sum_{i,j} \frac{y_\nu}{\Lambda}\ell_i(Y_\nu)_{ij}\nu_j^cH_u + \sum_{i,j} \frac{y_e}{\Lambda}\ell_i(Y_e)_{ij}e_j^cH_d + h.c. + \sum_{i,j} y_R\nu_i^c(Y_R)_{ij}\nu_j^c, \tag{1.14}$$

where q and ℓ are SU(2)$_L$ doublet fields, and f^c ($f = u,d,e,\nu$) are SU(2)$_L$ singlet fields. Here, in order to distinguish each Y_f from others, we have assigned U(1)$_X$ charges as $Q_X(f^c) = -x_f$, $Q_X(Y_f) = +x_f$ and $Q_X(Y_R) = 2x_\nu$. The VEVs of yukawaons are obtained from supersymmetric

(SUSY) vacuum conditions for a superpotential W. In the charged lepton sector, we assume

$$W_e = \lambda_e[\Phi_e\Phi_e\Theta_e] + \mu_e[Y_e\Theta_e] + W_\Phi, \qquad (1.15)$$

where the fields Φ_e and Θ_e have the U(1)$_X$ charges $\frac{1}{2}x_e$ and $-x_e$, respectively. Here and hereafter, for convenience, we denote a trace Tr$[A]$ of a matrix A as $[A]$ concisely. A SUSY vacuum condition $\partial W/\partial \Theta_e = 0$ leads to a bilinear mass relation

$$\langle Y_e \rangle = -\frac{\lambda_e}{\mu_e}\langle \Phi_e \rangle \langle \Phi_e \rangle. \qquad (1.16)$$

Therefore, the mass spectrum of the charged leptons are given by

$$K = \frac{m_e + m_\mu + m_\tau}{(\sqrt{m_e} + \sqrt{m_\mu} + \sqrt{m_\tau})^2} = \frac{v_1^2 + v_2^2 + v_3^2}{(v_1 + v_2 + v_3)^2} = \frac{[\langle \Phi_e \rangle \langle \Phi_e \rangle]}{[\langle \Phi_e \rangle]^2}, \qquad (1.17)$$

and

$$\kappa = \frac{\sqrt{m_e m_\mu m_\tau}}{(\sqrt{m_e} + \sqrt{m_\mu} + \sqrt{m_\tau})^3} = \frac{v_1 v_2 v_3}{(v_1 + v_2 + v_3)^3} = \frac{\det\langle \Phi_e \rangle}{[\langle \Phi_e \rangle]^3}, \qquad (1.18)$$

where $\langle \Phi_e \rangle = \mathrm{diag}(v_1, v_2, v_3)$. We refer the field Φ_e as an "ur-yukawaon". A case with $K = 2/3$ (an earlier attempt to give $K = 2/3$ has been done in Ref.[9]) is known as the charged lepton mass formula (1.1) which is excellently satisfied by the observed charged lepton masses (pole masses). However, note that masses which we deal with in the present paper are running masses, so that our goal is not $K^{pole} = 2/3$, but $K(\Lambda) = (2/3)(1 + \varepsilon(\Lambda))$ with $\varepsilon(\Lambda) \sim 10^{-3}$.

The term W_Φ in the superpotential (1.15) has been introduced in order to fix a VEV spectrum of $\langle \Phi_e \rangle$. The explicit form is given in the next section. The purpose of the present paper is heuristically to find a form of W_Φ which can simultaneously give reasonable values of $K(\Lambda)$ and $\kappa(\Lambda)$, and it is not to investigate an origin of such the form of W_Φ. The investigation of the theoretical origin will be our next step of the investigations.

2 VEV structure of the ur-yukawaon

In this paper, we consider a model with explicit U(1)$_X$ symmetry breakings of the order ε_{SB} which is negligibly small. A prototype of such a model has been discussed in Ref.[2]. We know that, in order to give $K = 2/3$, it is essential to bring interaction terms with a traceless part $\hat{\Phi}_e \equiv \Phi_e - \frac{1}{3}[\Phi_e]$ into the potential [7]. In the present model, we also consider a more general form of W_Φ,

$$W_\Phi = \varepsilon_{SB}\lambda_A \left\{ \alpha_1[\hat{\Phi}_e\hat{\Phi}_e\hat{Y}_e] + \alpha_2[\Phi_e\Phi_e\hat{Y}_e] + \alpha_3[\Phi_e\Phi_e Y_e] \right\}$$

$$+\varepsilon_{SB}\lambda_B \left\{ \beta_1[\hat{\Phi}_e\hat{\Phi}_e\hat{\Phi}_e] + \beta_2[\Phi_e\Phi_e\hat{\Phi}_e] + \beta_3[\Phi_e\Phi_e\Phi_e] \right\}, \qquad (2.1)$$

where $\hat{\Phi}_e$ and $\hat{Y}_e$ are **5** components of O(3) which are defined by

$$\hat{\Phi}_e \equiv \Phi_e - \frac{1}{3}[\Phi_e], \quad \hat{Y}_e \equiv Y_e - \frac{1}{3}[Y_e]. \qquad (2.2)$$

(Hereafter, we will drop the index "e", because we only deal with fields in the charged lepton sector in this paper.) The expression (2.1) is the most general form of $\Phi\Phi Y$ and $\Phi\Phi\Phi$ terms, because we know the following identical equations:

$$[\Phi\hat{\Phi}\hat{Y}] = \frac{1}{2}\left([\hat{\Phi}\hat{\Phi}\hat{Y}] + [\Phi\Phi\hat{Y}]\right), \tag{2.3}$$

$$[\hat{\Phi}\Phi Y] = [\hat{\Phi}\hat{\Phi}Y] + [\Phi\hat{\Phi}\hat{Y}] - [\hat{\Phi}\hat{\Phi}\hat{Y}], \tag{2.4}$$

and an equation with $Y \to \Phi$ in Eq.(2.3). We consider that the parameters λ_e, λ_A, λ_B, μ_e and ε_{SB} in the superpotential are "analogue" parameters which we can adjust continuously, while we assume that $\alpha_i(\Lambda)$ and $\beta_i(\Lambda)$ should be integers. We consider that the relations for $K(\Lambda)$ and $\kappa(\Lambda)$ are independent of these analogue parameters, and those should be described only by the "digital" parameters α_i and β_i. Hereafter, we will simply denote $K(\Lambda)$, $\kappa(\Lambda)$, $\ldots$, as K, κ, $\cdots$, because we discuss only the relations among quantities at $\mu = \Lambda$.

The terms given in Eq.(2.1) explicitly break not only the U(1)$_X$ symmetry but also the R symmetry, because we assign the R charges of Φ, Y and Θ to 0, 0 and 2, respectively. We assume that the value ε_{SB} is negligibly small. Therefore, the present model cannot break the supersymmetry spontaneously [10]. For the moment, we consider that the supersymmetry is unbroken, at least, in the yukawaon sector.

The VEV of the ur-yukawaon Φ is obtained from SUSY vacuum conditions as we show later. Then, we will obtain a cubic equation for eigenvalues v_i of $\langle\Phi\rangle$:

$$c_3\langle\Phi\rangle^3 + c_2\langle\Phi\rangle^2 + c_1\langle\Phi\rangle + c_0\mathbf{1} = 0, \tag{2.5}$$

which necessarily and sufficiently determines the eigenvalues of $\langle\Phi\rangle$. The coefficients c_a ($a = 0, 1, 2, 3$), in general, have the following relations:

$$\frac{c_2}{c_3} = -[\langle\Phi\rangle], \quad \frac{c_1}{c_3} = \frac{1}{2}\left([\langle\Phi\rangle]^2 - [\langle\Phi\rangle\langle\Phi\rangle]\right), \quad \frac{c_0}{c_3} = -\det\langle\Phi\rangle. \tag{2.6}$$

Therefore, we can express the quantities K and κ as follows:

$$K = 1 - 2\frac{c_1}{c_3}\frac{1}{[\langle\Phi\rangle]^2}, \tag{2.7}$$

$$\kappa = -\frac{c_0}{c_3}\frac{1}{[\langle\Phi\rangle]^3}, \tag{2.8}$$

by using the relations (2.6).

Prior to calculating SUSY vacuum conditions, it is useful to the expression (2.1) as follow:

$$W_\Phi = \varepsilon_{SB}\lambda_A\left\{a_0[\Phi\Phi Y] + a_1[\Phi][\Phi Y] + a_2[\Phi]^2[Y] + a_3[\Phi\Phi][Y]\right\}$$

$$+\varepsilon_{SB}\lambda_B\left\{b_0[\Phi\Phi\Phi] + b_1[\Phi\Phi][\Phi] + b_2[\Phi]^3\right\}, \tag{2.9}$$

where

$$a_0 = \alpha_1 + \alpha_2 + \alpha_3, \quad a_1 = -\frac{2}{3}\alpha_1, \quad a_2 = \frac{2}{9}\alpha_1, \quad a_3 = -\frac{1}{3}(\alpha_1 + \alpha_2), \tag{2.10}$$

$$b_0 = \beta_1 + \beta_2 + \beta_3, \quad b_1 = -\frac{1}{3}(3\beta_1 + \beta_2), \quad b_2 = \frac{2}{9}\beta_1. \tag{2.11}$$

Then, from a SUSY vacuum condition $\partial W/\partial Y = 0$, we obtain

$$\Theta = -\varepsilon_{SB}\frac{\lambda_A}{\mu}\left\{a_0\Phi\Phi + a_1[\Phi]\Phi + a_2[\Phi]^2\mathbf{1} + a_3[\Phi\Phi]\mathbf{1}\right\}. \tag{2.12}$$

From a SUSY vacuum condition $\partial W/\partial\Phi = 0$, we also obtain

$$\frac{\partial W}{\partial\Phi} = 0 = \lambda(\Phi\Theta + \Theta\Phi)$$

$$+\varepsilon_{SB}\lambda_A\left\{a_0(\Phi Y + Y\Phi) + a_1[\Phi]Y + a_1[\Phi Y]\mathbf{1} + 2a_2[Y][\Phi]\mathbf{1} + 2a_3[Y]\Phi\right\}$$

$$+\varepsilon_{SB}\lambda_B\left\{3b_0\Phi\Phi + 2b_1[\Phi]\Phi + b_1[\Phi\Phi]\mathbf{1} + 3b_2[\Phi]^2\mathbf{1}\right\}. \tag{2.13}$$

When the value of ε_{SB} is small, but it is not zero, by substituting Eqs.(1.16) and (2.12) for Eq.(2.13), we obtain the cubic equation (2.5) with coefficients

$$c_3 = 4a_0, \quad c_2 = (3a_1 - \xi b_0)[\Phi], \tag{2.14}$$

$$c_1 = 2(a_2[\Phi]^2 + 2a_3[\Phi\Phi] - \xi b_1[\Phi]^2), \tag{2.15}$$

$$c_0 = a_1[\Phi\Phi\Phi] + 2a_2[\Phi\Phi][\Phi] - \xi[\Phi](b_1[\Phi\Phi] + 3b_2[\Phi]^2), \tag{2.16}$$

where ξ is defined by

$$\xi = \frac{\lambda_B}{\lambda_A}\frac{\mu}{\lambda[\Phi]}, \tag{2.17}$$

and it is an "analogue" parameter at present. Note that the eigenvalues of $\langle\Phi\rangle$ are independent of the analogue parameters except for ξ (however, we will show later that ξ becomes a digital one).

By using a general formula for any 3×3 matrix

$$L \equiv \frac{[\Phi\Phi\Phi]}{[\Phi]^3} = 3\kappa + \frac{3}{2}K - \frac{1}{2}, \tag{2.18}$$

together with Eqs.(2.7) and (2.8), we can obtain

$$K = 1 - 2\frac{c_1}{c_3}\frac{1}{[\Phi]^2} = 1 - \frac{1}{a_0}(a_2 + 2a_3K - \xi b_1), \tag{2.19}$$

i.e.

$$K = \frac{1 - \frac{a_2}{a_0} + \xi\frac{b_1}{a_0}}{1 + 2\frac{a_3}{a_0}}, \tag{2.20}$$

and

$$\kappa = -\frac{c_0}{c_3} \frac{1}{[\Phi]^3} = -\frac{a_1}{4a_0} \left\{ 3\kappa + \frac{3}{2}K - \frac{1}{2} + 2\frac{a_2}{a_1}K - 2\xi\frac{b_1}{a_1}\left(K + 3\frac{b_2}{b_1}\right)\right\}, \tag{2.21}$$

i.e.

$$\kappa = \frac{1}{1 + \frac{3a_1}{4a_0}}\frac{a_1}{8a_0}\left\{1 - 3K - 4\frac{a_2}{a_1}K + 2\xi\frac{b_1}{a_1}\left(K + 3\frac{b_2}{b_1}\right)\right\}. \tag{2.22}$$

On the other hand, as seen in (2.6), in order to obtain a solution with $[\Phi] \neq 0$, from the relations (2.14), the parameter ξ must satisfy

$$0 = c_3 + \frac{c_2}{[\Phi]} = 4a_0 + 3a_1 - 3\xi b_0. \tag{2.23}$$

Therefore, the parameter ξ is not an adjustable parameters, and, now, it is expressed by the digital parameters α_i and β_i.

When we define a deviation of ε from the ideal value $K = 2/3$ as

$$K = \frac{2}{3}(1 + \varepsilon), \tag{2.24}$$

the quantities κ and K are expressed as follows: By using Eqs.(2.10) and (2.11) together with Eq.(2.23), the quantity κ is expressed as

$$\kappa = \frac{1}{54\rho}(1 - 12\rho\sigma + 2\omega\varepsilon), \tag{2.25}$$

where

$$\rho = -\left(1 + \frac{4a_0}{3a_1}\right) = \frac{\alpha_1 + 2\alpha_2 + 2\alpha_3}{\alpha_1}, \tag{2.26}$$

$$\sigma = -\left(1 + \frac{9b_2}{2b_1}\right)\frac{b_1}{b_0} = \frac{1}{3}\frac{\beta_2}{\beta_1 + \beta_2 + \beta_3}, \tag{2.27}$$

$$\omega = 5 + 6\rho\frac{b_1}{b_0} = 5 - 2\frac{\alpha_1 + 2\alpha_2 + 2\alpha_3}{\alpha_1}\frac{3\beta_1 + \beta_2}{\beta_1 + \beta_2 + \beta_3}. \tag{2.28}$$

Also, from Eq.(2.20), the deviation ε is expressed as

$$\varepsilon = \frac{3}{2}K - 1 = \frac{\Delta}{2(\alpha_1 + \alpha_2 + 3\alpha_3)(\beta_1 + \beta_2 + \beta_3)}, \tag{2.29}$$

where

$$\Delta = 2(\alpha_1 + 2\alpha_2 + 2\alpha_3)(\beta_3 - 2\beta_1) - (3\alpha_1 + 3\alpha_2 - \alpha_3)(\beta_1 + \beta_2 + \beta_3). \tag{2.30}$$

Since $K(\Lambda) = \frac{2}{3}(1 + \varepsilon(\Lambda)) = K^{pole}(1 + \varepsilon_K(\Lambda))$, i.e. $K^{pole} = \frac{2}{3}(1 + \varepsilon(\Lambda) - \varepsilon_K(\Lambda))$, we will search for a model which gives $\varepsilon(\Lambda) \simeq \varepsilon_K(\Lambda)$ numerically. (Note that this does not mean $\varepsilon(\mu) \simeq \varepsilon_K(\mu)$ at every energy scale μ.)

3 Toy model with a simpler structure

In this section, we investigate a model with a simpler form

$$W_\Phi = +\varepsilon_{SB}\lambda_A \left\{ \alpha_1 [\hat{\Phi}\hat{\Phi}\hat{Y}] + \alpha_3 [\Phi\Phi Y] \right\} + \varepsilon_{SB}\lambda_B \left\{ \beta_1 [\hat{\Phi}\hat{\Phi}\hat{\Phi}] + \beta_3 [\Phi\Phi\Phi] \right\}, \quad (3.1)$$

i.e. with a case $\alpha_2 = \beta_2 = 0$ in Eq.(2.1). Here, we assume that α_1, α_3, β_1 and β_3 are positive integers. (Note that this ansatz is only valid at $\mu = \Lambda$.) In the present model, we essentially have two parameters α_3/α_1 and β_3/β_1 for the two observables $K(\Lambda)$ and $\kappa(\Lambda)$, so that we cannot predict K and κ definitely. For large values α_i and β_i, the parameters α_3/α_1 and β_3/β_1 approximately behave as if they are continuous parameters. In order to exclude such the cases from consideration, we assume that α_i and β_i are not so large, e.g. they should be smaller than about 50. (When the parameters α_1 and α_3 are given by $\alpha_1 = kn_1$ and $\alpha_3 = kn_3$ (k, n_1 and n_3 are positive integers), we redefine α_1 and α_3 as $\alpha_1 = n_1$ and $\alpha_3 = n_3$. We also do such a re-definition for the parameters β_1 and β_3.)

In the present model (3.1), since $\beta_2 = 0$, the factor σ defined in (2.27) becomes $\sigma = 0$, so that the quantity κ is expressed simply as

$$\kappa = \frac{1}{54\rho}(1 + 2\omega\varepsilon). \quad (3.2)$$

Since $\varepsilon \sim 10^{-3}$, the value κ is almost determined by the parameter α_3/α_1:

$$\rho = 1 + 2\frac{\alpha_3}{\alpha_1}. \quad (3.3)$$

Since we search a case $\kappa \neq 0$, we consider $\alpha_1 \neq 0$. Since $\partial\kappa/\partial\alpha_1 > 0$, we can regard the value of α_1 as $\alpha_1 = 1$, which gives a next smaller value of κ following to $\kappa = 0$ at $\alpha_1 = 0$. Then, by neglecting the factor $2\omega\varepsilon \lesssim 10^{-2}$, the expression (3.2) prdicts

$$\kappa \simeq \frac{1}{54(1 + 2\alpha_3)}. \quad (3.4)$$

On the other hand, the deviation ε defined by (2.24) is given by

$$\varepsilon = \frac{\Delta}{2(3\alpha_3 + 1)(\beta_3 + \beta_1)}, \quad (3.5)$$

where

$$\Delta = (3\alpha_3 + 5)\beta_3 - (9\alpha_3 + 1)\beta_1. \quad (3.6)$$

The smallest value of ε for each α_3 can appear when $\Delta = 1$. (The equation $\Delta = 1$ is known as a Diophantine equation.) Our interest is whether there is a simpler set of $(\alpha_3, \beta_1, \beta_3)$ which can favorably give the observed values $K(\Lambda)$ and $\kappa(\Lambda)$.

Obviously, from Eq.(3.6), when $\beta_1 = \beta_1^0 \equiv 3\alpha_3 + 5$ and $\beta_3 = \beta_3^0 \equiv 9\alpha_3 + 1$, the case $\beta \equiv (\beta_1, \beta_3) = \beta_0 \equiv (\beta_1^0, \beta_3^0)$ gives $\varepsilon = 0$. [Exactly speaking, the lowest number of $\beta_0 \equiv (\beta_1^0, \beta_3^0)$

Table 2: Smallest value of $\varepsilon_{(n)}$ for $\beta_1 = \beta_1^0 - n$. (β_1^0, β_3^0) denotes a value which gives $\varepsilon = 0$ for a given value of α_3.

α_3	1	2	3	4	5	6	7	8	9	10
(β_1^0, β_3^0)	$(4,5)$	$(11,19)$	$(1,2)$	$(17,37)$	$(10,23)$	$(23,55)$	$(13,32)$	$(29,73)$	$(16,41)$	$(5,13)$
$\varepsilon_{(1)}$	$1/28$	$1/49$	—	$1/442$	$1/160$	$3/950$	$1/154$	$1/330$	$1/168$	$7/310$
(β_1, β_3)	$(3,4)$	$(10,18)$	—	$(16,35)$	$(9,21)$	$(22,53)$	$(12,30)$	$(28,71)$	$(15,39)$	$(4,11)$
$\varepsilon_{(2)}$	$1/10$	$1/70$	—	$1/208$	$1/72$	$1/152$	$2/143$	$1/4750$	$1/700$	$7/682$
(β_1, β_3)	$(2,3)$	$(9,16)$	—	$(15,33)$	$(8,19)$	$(21,51)$	$(11,28)$	$(27,68)$	$(14,36)$	$(3,8)$

which gives $\varepsilon = 0$ is given by $(\frac{1}{2}(3\alpha_3 + 5), \frac{1}{2}(9\alpha_3 + 1))$ for an odd number of α_3, and $\beta_0 = (3k + 2, 9k + 4)$ for $\alpha_3 = 3 \mod 7$ (i.e. for $\alpha_3 = 7k + 3$).] Since we want a case with $\varepsilon \neq 0$, we consider $\beta \neq \beta_0$. For $\beta > \beta_0$, i.e. for large values of β_1 and β_3, we can obtain any small value of ε. In order to exclude such the cases from consideration, we assume that the values β_1 and β_3 are not so large, and we confine our value $\beta = (\beta_1, \beta_3)$ as

$$(1, 1) \leq (\beta_1, \beta_3) < (3\alpha_3 + 5, 9\alpha_3 + 1). \tag{3.7}$$

At $\beta_1 = \beta_1^0$, ε takes $\varepsilon < 0$ for $1 \leq \beta_3 \leq \beta_3^0 - 1$ and $\varepsilon = 0$ for $\beta_3 = \beta_3^0$. At the next value $\beta_1 = \beta_1^0 - 1$, ε takes $\varepsilon < 0$ for $1 \leq \beta_3 \leq \beta_3^-$ and $\varepsilon > 0$ for $\beta_3^- + 1 \leq \beta_3 \leq \beta_3^0$. (For the explicit value of β_3^-, see Table 2. In Table 2, a value of β_3 in each $\varepsilon_{(n)}$ denotes $\beta_3 = \beta_3^- + 1$.) In general, for a given value $\beta_1 = \beta_1^0 - n$, the value of ε takes the smallest value $\varepsilon_{(n)}$ at $\beta_3 = \beta_3^- + 1$. We list the values $\varepsilon_{(1)}$ and $\varepsilon_{(2)}$ for typical cases of $\alpha = (\alpha_1, \alpha_3)$ and $\beta = (\beta_1, \beta_3)$ in Table 2. Note that the array $(\varepsilon_{(1)}, \varepsilon_{(2)}, \dots)$ are not always in order of magnitude. For example, the smallest value in $\alpha_3 = 4$ is $\varepsilon_{(6)} = 1/910$.

In Eq.(3.4), we have assumed $\alpha_1 = 1$ as the next value to $\alpha_1 = 0$ which gives $\kappa = 0$, because we consider $\kappa \neq 0$ in the realistic world. Similarly, we assume $\beta_1 = \beta_1^0 - 1$ as the next value to $\beta_1 = \beta_1^0$ which gives $\varepsilon = 0$, because we consider $\varepsilon \neq 0$ in the realistic world. (Since $\partial\varepsilon/\partial\beta_1 < 0$ against $\partial\kappa/\partial\alpha_1 > 0$, we take the next value as $\beta_1 = \beta_1^0 - 1$ against $\alpha_1 = 0 + 1$.) We assume that the physical value of ε is given by the smallest value $\varepsilon_{(1)}$ at $\beta = (\beta_1^0 - 1, \beta_3^- + 1)$. As seen in Table 2, of the smallest values of $\varepsilon_{(1)}$ for various values of α_3, the smallest one is

$$\varepsilon = \frac{1}{442} = 2.26244 \times 10^{-3}, \tag{3.8}$$

at $\alpha = (1, 4)$ and $\beta = (16, 35)$. The value (3.8) is in good agreement with the value of ε_K given in (1.12). (Exactly speaking, the value (3.8) is in excellent agreement with the value of ε_K at $\mu \sim 10^{11}$ GeV as seen in Table 1, although we consider $\Lambda \sim 10^{15}$ GeV from a naive estimate $m_\nu \sim \langle H_u^0 \rangle^2 / \Lambda$.) On the other hand, the choice $\alpha = (1, 4)$ predicts

$$\kappa = \frac{1}{486}(1 + 2\omega\varepsilon), \tag{3.9}$$

where $\omega = -609/51 = -11.94$. By inputting the value (3.8) into Eq.(3.9), we obtain

$$\kappa = 2.0756 \times 10^{-3} \times (1 - 0.0540) = 1.9464 \times 10^{-3}. \tag{3.10}$$

It is interesting that the value of κ in the limit of $\varepsilon = 0$ in Eq.(3.9)

$$\kappa|_{\varepsilon=0} = \frac{1}{486} = 2.058 \times 10^{-3}, \tag{3.11}$$

is in good agreement with the observed value $\kappa^{pole} = (2.0633 \pm 0.0001)$ correspondingly to the fact that K in the limit of $\varepsilon = 0$ gives $K = 2/3 = K^{pole}$. (The value $\kappa = 1/486$ has first been speculated from another scenario [11].) However, K and κ in the present section denote $K(\Lambda)$ and $\kappa(\Lambda)$, respectively, so that there is no theoretical ground for the conjecture that K and κ in the limit $\varepsilon = 0$ give K^{pole} and κ^{pole}.

4 Concluding remarks

First, we would like to re-emphasize that the quantities $K(\mu)$ and $\kappa(\mu)$, which are defined by Eqs.(1.2) and (1.3), respectively, are useful for investigating the charged lepton mass spectrum, because they are almost independent of the energy scale μ of the running masses.

In the present paper, against a conventional view, we have taken a standpoint that the observed relation $K^{pole} = 2/3$ is merely an approximate relation, and the ratio K is given by $K(\Lambda) = \frac{2}{3}(1 + \varepsilon(\Lambda))$ with $\varepsilon(\Lambda) \sim 10^{-3}$ (not $\varepsilon(\Lambda) = 0$) from the beginning. In order to understand $\varepsilon(\Lambda) \sim 10^{-3}$, on the basis of the so-called supersymmetric yukawaon model, we have investigated possible values of $K(\Lambda)$ and $\kappa(\Lambda)$. By taking a simpler form of the superpotential of the yukawaons (3.1) at $\mu = \Lambda$, and by assuming that the coefficient of each term in (3.1) is given by a lower positive integer at $\mu = \Lambda$, we have tried to speculate values of $K(\Lambda)$ and $\kappa(\Lambda)$. However, the present model (3.1) involves two parameters α_3/α_1 and β_3/β_1, so that the model cannot predict values of $K(\Lambda)$ and $\kappa(\Lambda)$ definitely. The model can predict only many candidates of the values $K(\Lambda)$ and $\kappa(\Lambda)$ (but as discrete values). Of the many candidates of the predicted values, a case of $\alpha = (1,4)$ and $\beta = (16,35)$ (i.e. $\alpha_3/\alpha_1 = 4$ and $\beta_3/\beta_1 = 2 + 3/16$) can give interesting results of $K(\Lambda)$ and $\kappa(\Lambda)$: $\varepsilon = 1/442 = 2.262 \times 10^{-3}$ and $\kappa = (1/486) \times (1 + 2\omega\varepsilon) = 2.058 \times 10^{-3} \times (1 - 0.054) = 1.95 \times 10^{-3}$, which are in fairly good agreement with our goal $\varepsilon_K(\Lambda) \simeq 2.4 \times 10^{-3}$ and $\kappa(\Lambda) \simeq 2.02 \times 10^{-3}$, respectively. Although, on trial in Sec.3, we have assumed that the nature chooses the smallest ε in the next β_1 to $\beta_1 = \beta_1^0$ (i.e. $\beta_1 = \beta_1^0 - 1$), and picked up the case of $\alpha = (1,4)$ and $\beta = (16,35)$, there is no theoretical ground for such a selection rule.

The present model is based on an effective theory which is valid at $\mu \leq \Lambda$. We have assumed that there is such the energy scale Λ at which the superpotential takes a simple form. A naive estimate of Λ leads to $m_\nu \sim \langle H_u^0 \rangle^2/\Lambda$, so that we consider $\Lambda \sim 10^{15}$ GeV. However, by taking VEV values of yukawaons and coefficients (coupling constants) suitably, we may consider $\Lambda \sim 10^{14-16}$ GeV. As we have emphasized in Sec.1, the effective coupling constants $Y_f^{eff} = (y_f/\Lambda)\langle Y_f \rangle$ evolve as in the standard model below the scale Λ. Although the renormalization group equations (RGEs) affect to the parts **1** and **5** of yukawaons with $(\mathbf{3} \times \mathbf{3})_S$ of O(3) separately, the effects are negligible in the present model. (For example, see Sec.6 in Ref.[12].)

Exactly speaking, a threshold effect at $\mu \sim \Lambda$ is also important in the present model. The singlet and 5-plet parts of the effective Yukawa interactions will separately receive such corrections in the present effective theory. However, in order to estimate the effect exactly, we

must know a full theory at $\mu > \Lambda$. Since we do not know such the full theory at present, in this paper, we are obliged to put an ansatz that the threshold effect is negligibly small. Our goal is to simultaneously understand the both values of $K(\Lambda)$ and $\kappa(\Lambda)$ at $\mu = \Lambda \sim 10^{14-16}$ GeV which are given in Table 1, even at the cost of such the accuracy in the observed fact $K^{pole} \simeq 2/3$.

The predicted values (3.8) and (3.10) are only valid within an approximation where we neglect such the threshold effect. In fact, the predicted value (3.8) at $\mu = \Lambda$ does not lead to $K^{pole} = 2/3$ in the exact meaning.

The superpotential terms W_Φ given in Eq.(3.1) spontaneously break the U(1)$_X$ symmetry, so that pseudo Goldstone bosons appear. An earlier estimate of such light yukawaons based on a prototype model of the present W_Φ is found in Ref.[2]. The conclusions in Ref.[2] that we have three massless scalars and three pseudo Goldstone bosons with masses of the order of $\varepsilon_{SB}\Lambda$ will be unchanged in the present model. For more details of physical meaning of the light bosons, we will discuss elsewhere.

In this paper, we have considered that the observed fact $K^{pole} = 2/3$ is accidental, and the model should give $K(\Lambda) = \frac{2}{3}(1 + \varepsilon(\Lambda))$ with $\varepsilon(\Lambda) \sim 10^{-3}$ from the beginning. However, the present model have failed to give such excellent coincidences about $K(\Lambda)$ and $\kappa(\Lambda)$ as the well-known relation $K^{pole} = 2/3$. In contrast to the present idea, recently, Sumino [5] has proposed a model which gives $K^{pole} = 2/3$ exactly. This idea is based on a point of view that the relation (1.1) for the pole masses is exactly valid, so that his standpoint is in opposite direction against the present approach. If the charged leptons are truly fundamental entities, the laws of the nature for the charged leptons should be beautifully simple, so that it seems to be likely that $K^{pole} = 2/3$ is exact as pointed out by Sumino. Even if Sumino's idea is true, the present approach based on the supersymmetric yukawaon model will still be promising for unified understanding of $K(\Lambda)$ and $\kappa(\Lambda)$, because the cubic equation (2.5) can completely determine the eigenvalues of $\langle\Phi\rangle$ (i.e. $K(\Lambda)$ and $\kappa(\Lambda)$ simultaneously). We hope that a simple requirement for W_Φ leads to simultaneous understanding of $K(\Lambda)$ and $\kappa(\Lambda)$ along a line suggested by a yukawaon model.

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